Due: Thursday October 12th, 2023, 1pm
Problems Due: 1, 4(a), 4(c), 5(c)

[AWH] Arfken, George B and Weber, Hans J and Harris, Frank E Mathematical methods for physicists: a comprehensive guide *Academic press*, 2011.

1. Exercise 7.2.2 in [AWH]
2. Exercise 7.2.13 in [AWH]
3. Exercise 7.2.18 in [AWH]
4. Let $a \in \mathbb{R}, b \in \mathbb{R}$. $f : [b, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function and $x : [0\infty) \to \mathbb{R}$ solve

$$x(t) = a + \int_0^t f(r, x(r))dr.$$

   Solve for x when

   (a) $a = b = 0$ and $f(t, x) = -3x + t + e^{-2t}$.
   (b) $a = \frac{1}{2}$, $b = 1$ and $f(t, x) = -2x + t^2 - t + 1$.
   (c) $f(t, x) = -\alpha x + \beta \exp(-\gamma t)$ with $\alpha, \gamma > 0$ and $\beta \in R$.

   In each case decide if there is a limiting equilibrium.

5. Let $r : [0, t] \to \mathbb{R}$ be a continuous function. Solve for $x : [0\infty) \to \mathbb{R}$ when

   (a) $\frac{d^2x}{dt^2}(t) + r(t) = 0$ for $t > 0$
   (b) $\frac{d^2x}{dt^2}(t) - x(t) + r(t) = 0$ for $t > 0$
   (c) $\frac{d^2x}{dt^2}(t) + x(t) + r(t) = 0$ for $t > 0$
   (d) $\frac{d^2x}{dt^2}(t) + x(t) + \cos(t) = 0$ for $t > 0$

Book keeping Exercises[1]

1. Let $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function and Lipschitz in x uniformly over t. Let $a \in \mathbb{R}$ and set $z_0(t) = a$ for all $t \geq 0$. Define for $n \geq 1, t \geq 0$,

$$z_n(t) = a + \int_0^t f(r, z_{n-1}(r))dr.$$

   (a) Fix $M > 0$ show that there is a $C_1, C_2 > 0$ such that

$$| z_{n+1}(t) - z_n(t) | \leq C_1 \frac{C_1 t^{n+1}}{n+1!}$$

   for all $0 \leq t \leq M$.

   (b) Conclude that $z_n(\cdot)$ converge uniformly for $t \in [0, M]$ to a function $x(\cdot)$ which satistfies

$$x(t) = a + \int_0^t f(r, x(r))dr$$

   for all $t \in [0, M]$.

   (c) Using Gronwall's lemma show that there is a unique x satisfying

$$x(t) = a + \int_0^t f(r, x(r))dr$$

   for all $t \in [0, M]$ and for any $M > 0$.

2. Let $m \in \mathbb{R}, p, q, x : [0, \infty) \to \mathbb{R}$ be such that x solves the Bernoulli ODE given by

$$\frac{dx}{dt} + px + qx^m = 0$$

   for $t > 0$. Suppose $z(t) = x^{1-m}(t)$ makes sense for all $t > 0$

   (a) Write down a ODE solved by z.

   (b) Write down a general solution z to the ODE.

   (c) Solve for x.

3. Let $p, q : [0, t] \to \mathbb{R}$ be continuous functions. Consider

$$V = \{x : [0, \infty) \to \mathbb{R} : \frac{d^2x}{dt^2}(t) + p(t)\frac{dx}{dt}(t) + q(t)x(t) = 0 \text{ for } t > 0.\}$$

   Show that V is a 2-dimensional vector space over $\mathbb{R}$.

4. Let $p, q, r : [0, t] \to \mathbb{R}$ be continuous functions.

   (a) Suppose $\phi : [0, \infty) \to \mathbb{R} \setminus \{0\}$ solves the homogeneous second order linear ODE

$$\frac{d^2x}{dt^2}(t) + p(t)\frac{dx}{dt}(t) + q(t)x(t) = 0$$

   for $t > 0$ then show that another linearly independent solution is given by

$$\psi(t) = \phi(t) \int_0^t \frac{1}{\phi(s)^2} \exp(-\int_0^s p(r)dr)ds$$

[1]These will not be part of deliverables as far as quiz or midterms or finals. The results will be used and the proofs will help in understanding of the subject.

(b) Suppose $\phi_i : [0\infty) \to \mathbb{R} \setminus \{0\}$ are two l.i. solutions of the homogeneous second order linear ODE

$$\frac{d^2x}{dt^2}(t) + p(t)\frac{dx}{dt}(t) + q(t)x(t) = 0$$

for $t > 0$ then show that

$$\psi(t) = c_1\phi_1(t) + c_2\phi_1(t) - \int_0^t \frac{\phi_1(s)\phi_2(t) - \phi_2(s)\phi_1(t)}{\phi_1(s)\phi_2'(s) - \phi_2(s)\phi_1'(s)} r(s)ds$$

solves

$$\frac{d^2x}{dt^2}(t) + p(t)\frac{dx}{dt}(t) + q(t)x(t) + r(t) = 0$$

for $t > 0$