**Due: Thursday October 17th, 2023, 3:30pm**

1. *(Frobenius Method)* Let $R > 0$ and $p, q, x : (-R, R) \to \mathbb{R}$ solve the differential equation

$$\frac{d^2x}{dt^2}(t) + p(t)\frac{dx}{dt}(t) + q(t)x(t) = 0, \tag{1}$$

for $t \in (-R, R)$. Assume that $tp$ and $t^2q$ are analytic functions in $(-R, R)$, i.e.

$$tp(t) = \sum_{k=0}^{\infty} p_k t^k, \qquad \text{and } t^2q(t) = \sum_{k=0}^{\infty} q_k t^k,$$

for $t \in (-R, R)$ with $p_k, q_k \in \mathbb{R}$ for $k \geq 0$. Suppose

$$x(t) = t^\nu \sum_{k=0}^{\infty} a_k t^k$$

for some $\nu \in \mathbb{R}$, $a_0 \neq 0$, and $a_k \in \mathbb{R}$ for $k \geq 1$ is a solution to (1) for $0 < t < R$.

(a) Show that[1] for $n = 0, 1, 2, \ldots$

$$a_n\left[(\nu+n)(\nu+n-1) + (\nu+n)p_0 + q_0\right] + \sum_{k=0}^{n-1} a_k\left[(\nu+k)p_{n-k} + q_{n-k}\right] = 0$$

and consequently that $\nu$ should a solution to

$$\nu(\nu-1) + \nu p_0 + q_0 = 0. \tag{2}$$

(2) is called the *Indicial equation*.

(b) *(Frobenius Series)* Assume that the Indicial equation has two real roots $\nu_1, \nu_2$.

i. Let $\nu_1 \geq \nu_2$. Then show that

$$x(t) = t^{\nu_1} \sum_{k=0}^{\infty} a_k t^k$$

is a solution for $0 < t < R$, with all the $a_k$ being determined once $a_0 \neq 0$ and $\nu_1$ are given.

ii. Let $\nu_1 \geq \nu_2$ and suppose $\nu_1 - \nu_2$ is not an integer. Then show that

$$x(t) = t^{\nu_2} \sum_{k=0}^{\infty} b_k t^k$$

is a solution for $0 < t < R$ with all the $b_k$ being determined once $b_0 \neq 0$ and $\nu_1$ are given.

[1] *Hint: Use via the Cauchy Product*

$$p(t)\frac{dx}{dt}(t) = t^{\nu-2}\sum_{n=0}^{\infty}\left((\nu+n)p_0a_n + \sum_{k=0}^{n-1}(\nu+k)p_{n-k}a_k\right)t^n, \qquad q(t)x(t) = t^{\nu-2}\sum_{n=0}^{\infty}\left(q_0a_n + \sum_{k=0}^{n-1}q_{n-k}a_k\right)t^n$$