Due: Thursday October 19th, 2023, 3:30pm

1. *(Euler's equation)* Let $p, q \in \mathbb{R}$, and $x : (0, \infty) \to \mathbb{R}$ solve the differential eq(uation

$$t^2 \frac{d^2x}{dt^2}(t) + tp\frac{dx}{dt}(t) + qx(t) = 0, \tag{1}$$

for $t \in (0, \infty)$.

   (a) Provide a plausible reason why solution x to (1) may not be a real analytic.

   (b) For $s \in \mathbb{R}$ set $y(s) = x(e^s)$. Show that y solves the the differential equation

$$\frac{d^2y}{ds^2}(s) + (p-1)\frac{dy}{ds}(s) + qy(s) = 0, \tag{2}$$

   for $s \in \mathbb{R}$.

      i. For a given p, q, find the two linearly indepdendent solutions for ODE (2).

      ii. Using (i) for given p, q, find the two linearly indepdendent solutions for ODE (1).

2. *(Bessel's Equation)* Let $\lambda \geq 0$ and $x : [0, \infty) \to \mathbb{R}$ solve the differential equation

$$^2\frac{d^2x}{dt^2}(t) + t\frac{dx}{dt} + (t^2 - \lambda^2)x(t) = 0, \tag{3}$$

for $t \in (0, \infty)$.

   (a) *(Bessel Function)* For any fixed $\lambda \geq 0$ show that[1] the general solution x of (3) is of the form

$$x_\lambda(t) = t^\lambda \sum_{n=0}^{\infty} a_n t^n,$$

   with $a_0 \neq 0$ and find a recurrence relation for a_n.

   (b) For[2] $\lambda = 0, 1$ show that the Bessel functions of order $0, 1$ (i.e. solutions to the ODE) are given by

$$x_0(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{2^{2n}(n!)^2},$$

   and

$$x_1(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2^{2n+1}n!(n+1)!}$$

   (c) Can you write down a series expansion for Bessel function of order λ ?

[1] Use Frobenius Method

[2] Hint: Set $a_{-1} = 0$ and solve recurrence