Problem Set 1: Date Due: October 4, 2020

1. Consider the operator

$$\widehat{X}(t) = \int d^3r\, \Psi^\dagger(t,\vec{r}\,)\, x\, \Psi(t,\vec{r}\,)$$

in the second quantized Schrodinger field theory describing a system of identical bosons. Ψ denotes the field variable and $\Psi^\dagger$ denotes its hermitian conjugate. x denotes the first component of $\vec{r}$, i.e. $\vec{r} = (x, y, z)$.

   (a) Let $\vec{a}$ and $\vec{b}$ be two vectors:

$$\vec{a} = (a_x, a_y, a_z), \qquad \vec{b} = (b_x, b_y, b_z)$$

   Show that the states

$$\Psi^\dagger(t,\vec{a}\,)|0\rangle, \qquad \Psi^\dagger(t,\vec{a}\,)\Psi^\dagger(t,\vec{b}\,)|0\rangle$$

   are eigenstates of $\widehat{X}(t)$. Find the eigenvalues. 5

   (b) Now consider the operator

$$\widehat{P}(t) = -i\hbar \int d^3r\, \Psi^\dagger(t,\vec{r}\,)\, \frac{\partial \Psi(t,\vec{r}\,)}{\partial x}$$

   in the same theory. Find its eigenstates within one and two particle sectors and write down the eigenvalues. 5

Problem Set 2: Date Due: October 11, 2020

We have set $\hbar = 1$, $c = 1$

1. Consider a scalar field theory described by the action

$$S = \int dt \int d^3x \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{1}{2}m^2\phi^2\right]$$

   However, the x^1 direction, instead of being infinite in extent, is taken to be periodic with period L. In effect this means that the field ϕ as well as its conjugate momenta must be periodic under $x^1 \to x^1 + L$.

   Find the energy eigenstates and the corresponding eigenvalues. (Energy is to be measured relative to the vacuum of the theory, *i.e.* the vacuum by definition must have zero energy.)

   Do the states have interpretation as a set of particles? In particular what interpretation do the finite energy states have in the $L \to 0$ limit?

   10

2. Consider a scalar field theory described by the action

$$S = \int dt \int d^3x \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{\alpha}{2}(x^1)^2\phi^2 + \frac{\sqrt{\alpha}}{2}\phi^2 - \frac{1}{2}m^2\phi^2\right]$$

   where α is a constant. Find the energy eigenstates and the corresponding eigenvalues. (Energy is to be measured relative to the vacuum of the theory, *i.e.* the vacuum by definition must have zero energy.)

   Do the states have interpretation as a set of particles? In particular what interpretation do the finite energy states have in the $\alpha \to \infty$ limit?

   10

Problem Set 3: Date Due: October 18, 2020

1. Consider a classical field theory of a scalar field ϕ with the action:

$$S = K \int d^4x \, \phi \, \Box^2 \, \phi$$

where K is a constant,

$$\Box = \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \, ,$$

and $\Box^2 \equiv \Box \, \Box$.

Find expressions for the total energy and total momentum of the system. 10

2. Consider a classical particle moving in one dimension, described by the action

$$S = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \, F(t-t') \, q(t) \, q(t') \tag{1}$$

where t denotes time, q denotes the coordinate of the particle, and $F(t-t')$ is some given function of $(t-t')$ with the property that

$$F(t-t') = F(t'-t) \, .$$

Furthermore suppose that $F(t-t')$ goes to zero rapidly as $|t-t'| \to \infty$.

(a) Find the equation of motion of the particle by requiring that $\delta S = 0$ under any variation $\delta q(t)$ that vanishes as $t \to \pm\infty$. 3

(b) Show that the problem has time translation symmetry under which

$$q(t) \to \tilde{q}(t) = q(t+c)$$

for any constant c. 2

(c) Find an expression for the conserved quantity $Q(t)$ associated with time translation symmetry. This expression must be such that the dependence of the quantity $Q(t)$ at time t on $q(t')$ will fall off rapidly as $|t-t'|$ becomes large. 5

P.T.O.

(d) Show that $Q(t)$ defined this way is conserved when equation of motion is satisfied. 5

(e) Find a choice of $F(t-t')$ for which the action given in eq.(1) reduces to the action of a harmonic oscillator

$$S = \int dt \left(\frac{1}{2} m \left(\frac{dq}{dt} \right)^2 - \frac{1}{2} K q^2 \right)$$

where m and K are constants. Check that the expression for the conserved quantity that you have found reduces to the standard expression for the total energy of the harmonic oscillator for this choice of $F(t-t')$. 5

Note: You are allowed to use δ function and its derivatives in constructing F.

Note: You can treat this system using the same method that we developed for classical field theory, by regarding this as a field theory in 0 space and 1 time dimensions, interpreting $q(t)$ as the field.

Problem Set 4: Date Due: October 25, 2020 marks: 10+10

1. Consider the classical field theory describing the Schrodinger equation of a free non-relativistic particle. The corresponding action is

$$S = \int dt \int d^3x \, \Psi^*(\vec{x}, t) \left[i\frac{\partial}{\partial t} + \frac{1}{2m}\vec{\nabla}^2 \right] \Psi(\vec{x}, t)$$

   Since this describes a non-relativistic particle, naively we would expect that the action should be invariant under a Galilean transformation of the form:

$$\widetilde{\Psi}(\vec{x}, t) = \Psi(\vec{x} - \vec{v}t, t)$$

   where $\vec{v}$ is some fixed vector. Check that this is *not* a symmetry of the action.

   Next try to modify the transformation to:

$$\widetilde{\Psi}(\vec{x}, t) = e^{i\phi(\vec{v}, \vec{x}, t)} \, \Psi(\vec{x} - \vec{v}t, t)$$

   where $\phi(\vec{v}, \vec{x}, t)$ is a phase factor. Show that for an appropriate choice of $\phi(\vec{v}, \vec{x}, t)$ we get a symmetry.

   Next take $\vec{v}$ to be small to construct an infinitesimal symmetry transformation, and find the conserved charges associated with this symmetry.

2. Consider the non-relativistic Schrodinger field theory in the presence of a central potential $V(\vec{x})$ where V depends only on the magnitude $|\vec{x}|$ of $\vec{x}$. The corresponding action is:

$$S = \int dt \int d^3x \, \Psi^*(\vec{x}, t) \left[i\frac{\partial}{\partial t} + \frac{1}{2m}\vec{\nabla}^2 - V(\vec{x}) \right] \Psi(\vec{x}, t)$$

   Show that the action is invariant under the transformation:

$$\widetilde{\Psi}(\vec{x}, t) = \Psi(R\vec{x}, t)$$

   where R is a 3×3 rotation matrix.

   Now consider the infinitesimal version of this symmetry where:

$$R_{ij} = \delta_{ij} + \epsilon\omega_{ij}$$

   ω being an arbitrary 3×3 anti-symmetric matrix and ϵ being an infinitesimal parameter. Find expressions for the conserved charges associated with these symmetry transformations.

Problem Set 5: Date Due: November 15, 2020

1. Consider the harmonic oscillator with Hamiltoniam:
$$\frac{1}{2}p^2 + \frac{1}{2}\kappa^2 q^2$$
where q and p are conjugate coordinate and momentum variables. Let $|0\rangle$ denote the ground state of the theory. Calculate
$$\Delta_F(t_1, t_2) = \langle 0|T(q(t_1)\, q(t_2))|0\rangle$$
where T denotes time ordering. 5

2. Now add to the above Hamiltonian the interaction term:
$$H_{\rm int} = \frac{\lambda}{4!}\, q^4$$
Let $|\Omega\rangle$ denote the ground state of this theory. Calculate
$$G(t_1, t_2) = \langle \Omega|T(q(t_1)\, q(t_2))|\Omega\rangle$$
to order λ. Please try to carry out any integral that you might encounter, paying due attention to the $i\epsilon$ prescription and the order of limits. 5

Optional problems

1. Extend the calculation in problem 2 above to order λ^2.

2. Consider the action:
$$S = \int d^4\, x \left[-\frac{1}{2}\, \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\, \phi^4 \right]$$
Note that the coefficient of ϕ^2 has opposite sign compared to what we have considered before. In this case the perturbation theory we have developed does not work since the unperturbed Hamiltonian, containing quadratic terms in the fields and the conjugate momenta, does not

have energy bounded from below. Instead we use the fact that the potential term

$$V(\phi) = -\frac{1}{2}m^2\,\phi^2 + \frac{\lambda}{4!}\,\phi^4$$

has a pair of minima at

$$\phi = \pm a, \qquad a = \sqrt{6\,m^2/\lambda}\,.$$

Since the two minima are equivalent (due to $\phi \to -\phi$ symmetry of the action) we can focus on the minima at $\phi = a$. We now define:

$$\chi = \phi - a$$

and express the original action in terms of χ:

$$S = \int d^4\,x\,\left[-\frac{1}{2}\,\eta^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \frac{1}{2}m^2(\chi+a)^2 - \frac{\lambda}{4!}\,(\chi+a)^4\right]\,.$$

(a) Show that the Hamiltonian derived from the action, when expanded in powers of χ has the form

$$H = H_0 + H_{int}$$

where H_0 contains terms quadratic in χ, and has the correct sign of the χ^2 term so that it is bounded from below. H_{int} can be treated using perturbation theory of the kind we have discussed.

In particular, check that the Hamiltonian has no term linear in χ. If we had such terms then we shall not have regular perturbation expansion of the type we discussed. For this it is important that a corresponds to the minimum of $V(\phi)$. (You can throw away the constant term in H).

(b) Now note that the original action has a symmetry under $\phi \to \widetilde{\phi} = -\phi$. Using $\phi = a + \chi$ and $\widetilde{\phi} = a + \widetilde{\chi}$, this translates to the transformation

$$\chi \to \widetilde{\chi} = -2\,a - \chi$$

Show that this symmetry does not exist in the Green's function, e.g.

$$\langle\Omega|T(\widetilde{\chi}(x_1)\,\widetilde{\chi}(x_2))|\Omega\rangle \neq \langle\Omega|T(\chi(x_1)\,\chi(x_2))|\Omega\rangle$$

It is enough to show that the leading term in the expansion in powers of λ fails to respect this symmetry.

This demonstrates that the $\phi \to -\phi$ symmetry is spontaneously broken.

Problem Set 6: Date Due: December 13, 2020

1. Consider a scalar field theory with action:

$$-\frac{1}{2}\int d^4x\,\left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] - \frac{\lambda}{4!}\int d^4x\,\phi^4(x)$$

where λ is a constant.

(a) Calculate the wave-function renormalization constant Z and the physical mass m_p to order λ. 5

(b) Calculate the complete S-matrix element $S(\vec{p}_1, \vec{p}_2; \vec{k}_1, \vec{k}_2)$ for a pair of incoming particles of momenta $\vec{k}_1$ and $\vec{k}_2$ to scatter into a pair of outgoing particles of momenta $\vec{p}_1$ and $\vec{p}_2$, to order λ^2. You can assume that $\vec{p}_1 \neq \vec{k}_1, \vec{k}_2$ and $\vec{p}_2 \neq \vec{k}_1, \vec{k}_2$. 10

You do not need to carry out any loop momentum integration in either of these problems. You can give the result as integrals over loop momenta.

Optional problem

1. Consider a (3+1) dimensional quantum field theory of a single scalar field ϕ with action

$$\int d^4x\left[-\frac{1}{2}\eta^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - \lambda\,\phi^2\vec{\nabla}\phi\cdot\vec{\nabla}\phi\right]$$

where $\vec{\nabla}\phi$ denotes gradient in the spatial directions.

Write down the expression for

$$\langle\Omega|T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))|\Omega\rangle$$

to order λ^2. Here $|\Omega\rangle$ denotes the ground state of the hamiltonian.

In this calculation you can only use the contribution from the connected diagrams. Also you do not need to evaluate any integral, but express the result as integrals.

Problem set 7: Date due: December 20, 2020

1. Consider the lagrangian of a scalar field theory with ϕ^3 interaction:

$$L = -\frac{1}{2} \int d^3r \, \left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right] - \frac{g}{3!} \int d^3r \, \phi^3(t, \vec{r})$$

where g is a constant. The corresponding Hamiltonian has energy unbounded from below, but we shall ignore this problem and work in a perturbation expansion in g.

(a) Calculate the complete S-matrix element $S(\vec{p}_1, \vec{p}_2; \vec{k}_1, \vec{k}_2)$ for a pair of incoming particles of momenta $\vec{k}_1$ and $\vec{k}_2$ to scatter into a pair of outgoing particles of momenta $\vec{p}_1$ and $\vec{p}_2$, to order g^2. 8

(a) Using this S-matrix element, calculate the differential cross section $\left(\frac{d\sigma}{d\Omega}\right)_{cm}$ in the center of mass frame for two incoming particles of momenta $\vec{k}$ and $-\vec{k}$ to go into two outgoing particles of momenta $\vec{p}$ and $-\vec{p}$, as a function of the angle θ between $\vec{k}$ and $\vec{p}$. 7

Here $d\Omega = \sin\theta d\theta d\phi$ is the solid angle projected at the origin by the angular range between θ and $\theta + d\theta$ and between ϕ and $\phi + d\phi$. θ, ϕ are the polar and azimuthal angles in the spherical polar coordinate system.

Optional problem

1. Consider the lagrangian of a free scalar field theory in (3+1) dimensions:

$$L = -\frac{1}{2} \int d^3r \, \left[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right]$$

Now introduce a new field χ through the relation:

$$\phi = \chi + \frac{1}{2}\lambda\chi^2$$

(a) Express the lagrangian in terms of χ and its derivatives.

(b) Express the momentum Π conjugate to χ in terms of χ and its derivatives.

(c) Construct the Hamiltonian in terms of χ and Π.

(d) Calculate the three and four point correlation functions:

$$\langle 0|T(\chi(x_1)\chi(x_2)\chi(x_3))|0\rangle$$

and

$$\langle 0|T(\chi(x_1)\chi(x_2)\chi(x_3)\chi(x_4))|0\rangle$$

to order λ and λ^2 respectively.

(e) Show that although the four point correlation function is non-zero, the S-matrix $S(\vec{p}_1, \vec{p}_2|\vec{k}_1, \vec{k}_2)$ for the χ-particle vanishes.

Problem set 8: Date due: December 28, 2020

1. Consider free electrodynamics with action

$$S = -\frac{1}{4} \sum_{\mu,\nu=0}^{3} \int d^4x \, F_{\mu\nu}(x) F^{\mu\nu}(x) \, , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

In this theory calculate

$$\int d^4x \, e^{-ik\cdot(x-y)} \, \langle 0 | T \left(F_{\mu\nu}(x) F_{\rho\sigma}(y) \right) | 0 \rangle$$

where T denotes time ordering and $|0\rangle$ denotes the ground state of the theory.

Please simplify the expression to the extent possible. The final answer should be a function of k. 10

Problem set 9: Date due: January 16, 2020

1. Consider the field theory with action

$$S = \sum_{\mu,\nu=0}^{3} \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - eA_\mu \bar{\psi}\gamma^\mu\psi \right]$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Calculate the S-matrix element, to order e^2, for an $e^-\gamma$ pair of momentum $\vec{p}_1$ and $\vec{k}_1$ and spin states r_1 and a_1, to go into an $e^-\gamma$ pair of momentum $\vec{p}_2$ and $\vec{k}_2$ and spin states s_2 and a_2, respectively. Here e^- denotes the particle associated with the field ψ and γ denotes the particle associated with the field A_μ. 15

Problem set 10: Date due: January 20, 2020

1. Consider the field theory with action

$$S = \sum_{\mu,\nu=0}^{3} \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - eA_\mu \bar{\psi}\gamma^\mu \psi \right]$$

   where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

   (a) Calculate the S-matrix element, to order e^2, for an $e^- e^+$ pair of momentum $\vec{p}_1$ and $\vec{p}_2$ and helicities r_1 and r_2, to go into an $e^- e^+$ pair of momentum $\vec{p}_1$ and $\vec{p}_2$ and helicities s_1 and s_2, respectively. Here e^- and e^+ denote respectively the particle and anti-particle associated with the field ψ. 15

   (b) Calculate the total cross section of this process by integrating over the final state momenta and summing over the spin states of the final state particles. Try to simplify your result using identities involving u_α's and v_α's. 15

Optional problem

1. Consider a field theory involving a scalar field ϕ and a Dirac field ψ, described by the action:

$$S = \int dt \int d^3x \left[-\frac{1}{2}\, \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} M^2 \phi^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + g\bar{\psi}\psi\phi \right]$$

   Calculate the S-matrix element for two scalar particles, carrying momenta $\vec{p}_1$ and $\vec{p}_2$ to go into an $e^- e^+$ pair carrying momenta $\vec{k}_1$ and $\vec{k}_2$, and helicities s_1 and s_2 respectively, to order g^2. (Here e^- and e^+ represent particle and anti-particle associated with the Dirac field ψ.)