8/9/06

Non-abelian gauge theories

Draw analogy with QED.

Recall :- QED action is

$$S = \int d^4x \, \mathcal{L}$$

where $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} \{ i\gamma^\mu (\partial_\mu - ieA_\mu) - m \} \Psi$

such that $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Dirac indices aren't written explicitly

Symmetry of $\mathcal{L}$

① Global symmetry :-

$$\Psi(x) \to e^{ie\lambda} \Psi(x), \quad A_\mu(x) \to A_\mu(x)$$

λ → constant

This is valid even in the absence of gauge fields A_μ & their coupling to Ψ.

② Local symmetry / gauge symmetry :-

$$\Psi(x) \to e^{ie\lambda(x)} \Psi(x), \quad A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x)$$

Here gauge field is essential to make $\mathcal{L}$ invariant

→ gauge fields are essential.

Not a symmetry of free fermion action.

But it is a symmetry of the

Maxwell action without fermions.

We will be led to postulating the presence of gauge fields if we start with free Fermionic action & then take $\lambda(x)$ & demand it to be a sym. of $\mathcal{L}$

Define covariant derivative :-

$$D_\mu \psi = \partial_\mu \psi - ie A_\mu \psi$$

(Reason for defining this) → Under local gauge transformation

$$D_\mu \psi \longrightarrow e^{ie\lambda(x)} D_\mu \psi$$

<u>Proof</u> :-

$$D_\mu \psi \to \partial_\mu \left(e^{ie\lambda(x)} \psi\right) - ie\left(A_\mu + \partial_\mu \lambda\right)\left(e^{ie\lambda(x)} \psi\right)$$

$$= e^{ie\lambda(x)} \partial_\mu \psi + ie\, \partial_\mu \lambda(x) e^{ie\lambda(x)} \psi - ie A_\mu e^{ie\lambda(x)} \psi - ie\, \partial_\mu \lambda\, e^{ie\lambda(x)} \psi$$

$$= e^{ie\lambda(x)} \left(\partial_\mu \psi - ie A_\mu \psi\right)$$

Though $\partial_\mu \psi$ transforms in a complicated way, $D_\mu \psi$ transforms in a nice fashion — that's why it is called the covariant deriv.

Note → For $e^{ie\lambda} \to ?$

We have $(\partial_\nu - ieA_\mu) e^{ie\lambda(x)} (ie\partial_\nu \lambda - ieA_\nu)$

$= e^{ie\lambda(x)} (ie\partial_\mu \lambda - ieA_\mu)(ie\partial_\nu \lambda - ieA_\nu)$

$= (ie) e^{i\lambda} \left(\partial_\mu \partial_\nu \lambda - (\partial_\mu \lambda) A_\nu - A_\mu (\partial_\nu \lambda) + A_\mu A_\nu\right)$

⇓ sym. in μ, ν.

$\therefore F_{\mu\nu} e^{ie\lambda} = 0$

$$[D_\mu, D_\nu]\psi = D_\mu D_\nu \psi - D_\nu D_\mu \psi$$

$$= \left(\partial_\mu - ie A_\mu(x)\right)\left(\partial_\nu - ie A_\nu(x)\right)\psi(x)$$

$$- \nu \leftrightarrow \mu$$

$$= -ie F_{\mu\nu}(x) \psi(x)$$