Statistical Mechanics Exam: Part I

1. Consider a cylinder of radius a, kept vertically, containing N molecules of a monatomic gas. m is the mass of each atom, g is the acceleration due to gravity, and T is the temperature of the gas. The upper end of the cylinder is closed with a piston of weight w. In the equilibrium situation the piston is at a height h above the lower end of the cylinder. The height h is large so that the effect of the gravitational pull on the atoms cannot be ignored. The temperature is sufficiently large so that we can use classical statistical mechanics.

   a) Find an expression for the free energy F of the system as a function of a, N, m, g, h.

   b) Using the result of part a), and the equation $dF = -dW - SdT$ where dW is the work done by the system, and S is the entropy of the system, calculate the net force exerted by the gas on the piston, i.e. calculate the weight w in terms of a, N, m, g, h.

   c) Calculate the equilibrium density of atoms at a height x above the bottom of the cylinder.

   d) Using the result of part c), calculate the pressure at the top of the cylinder and the force exerted by the gas on the piston. Compare with the result of part a).

2. Consider a two dimensional ideal monatomic Bose gas containing N atoms each of mass m, confined in an area A with the geometry of a square, and periodic boundary condition in both directions. Find the temperature T at which the fraction f (f is small compared to 1 but fN is large compared to 1) of the total number of particles will be in the ground state. T should be determined in terms of N, A and m. (Note that you need to do this calculation for large but finite A.)

3. Consider a lattice containing two rows of sites as follows

   x x x x x x x x $\cdots$ x x x x x x x x

   x x x x x x x x $\cdots$ x x x x x x x x

   Each row contains $2N$ sites, with the $(2N+1)$th site identified with the first site. To the ith site in the upper row we associate a variable

s_i, and to the ith site in the lower row we associate a variable σ_i. s_i and σ_i take values ± 1. The total energy of the system for a given configuration of spins is given by:

$$E(\{s_i\}, \{\sigma_i\}) = -J \sum_{i=1}^{2N} (\sigma_i s_{i+1} + s_i \sigma_{i+1})$$

Find an expression for the free energy and the specific heat of this system.

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Fermions with spins

$\longrightarrow$ $(2s+1)$ degenerate states for every $\vec{p}$. [in absence of mag. field]

$$\ln \mathcal{Q} = (2s+1)\frac{V}{h^3}\int d^3p \ln\left(1+ze^{-\beta e(\vec{p})}\right)$$

$$N = (2s+1)\frac{V}{h^3}\int d^3p \frac{ze^{-\beta e(\vec{p})}}{1+ze^{-\beta e(\vec{p})}}$$

$$\frac{PV}{NkT} = V/N \frac{\partial}{\partial V}(\ln \mathcal{Q}) = \frac{f_1(z)}{f_2(z)}$$ same functions (as before)

$$\frac{N}{2s+1} = \frac{V}{h^3}\int d^3p \frac{ze^{-\beta e(\vec{p})}}{1+ze^{-\beta e(\vec{p})}}$$

$\longrightarrow$ same eqn. as in the earlier case with N replaced by $\frac{N}{2s+1}$.

Ideal Bose gas

$$\ln \mathcal{Q} = -\sum_{\vec{p}} \ln\left(1-ze^{-\beta e(\vec{p})}\right)$$

$$\overline{N} = \sum_{\vec{p}} \frac{ze^{-\beta e(\vec{p})}}{1-ze^{-\beta e(\vec{p})}}, \quad \overline{n(\vec{p})} = \frac{ze^{-\beta e(\vec{p})}}{1-ze^{-\beta e(\vec{p})}}$$

$\overline{n(\vec{p})} \geq 0$ for every $\vec{p}$

$\xrightarrow[\text{us to have}]{\text{force}} z \leq 1$

[why is this happening? $\rightarrow$ bcos $\sum_k \{ze^{-\beta e(\vec{p})}\}^k = \frac{1}{1-ze^{-\beta e(\vec{p})}}$ only when $z \leq 1$]